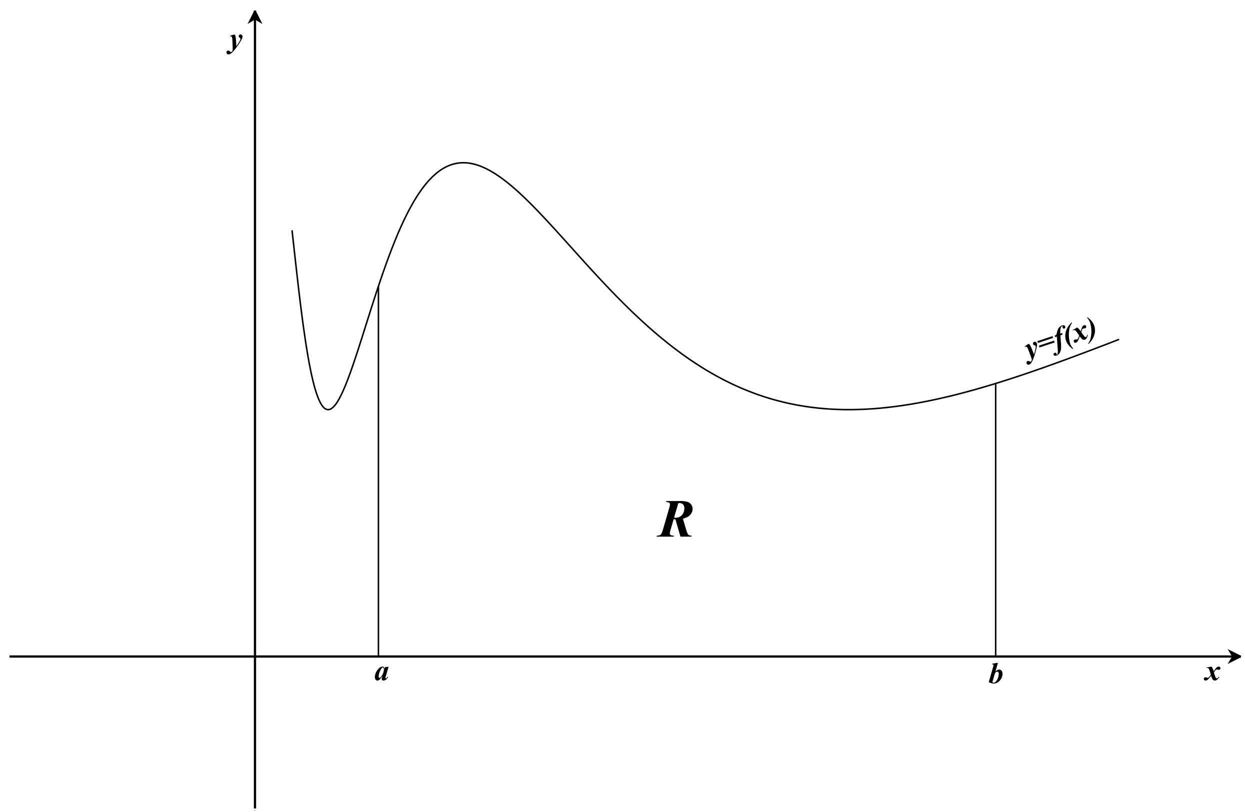
The Definite Integral

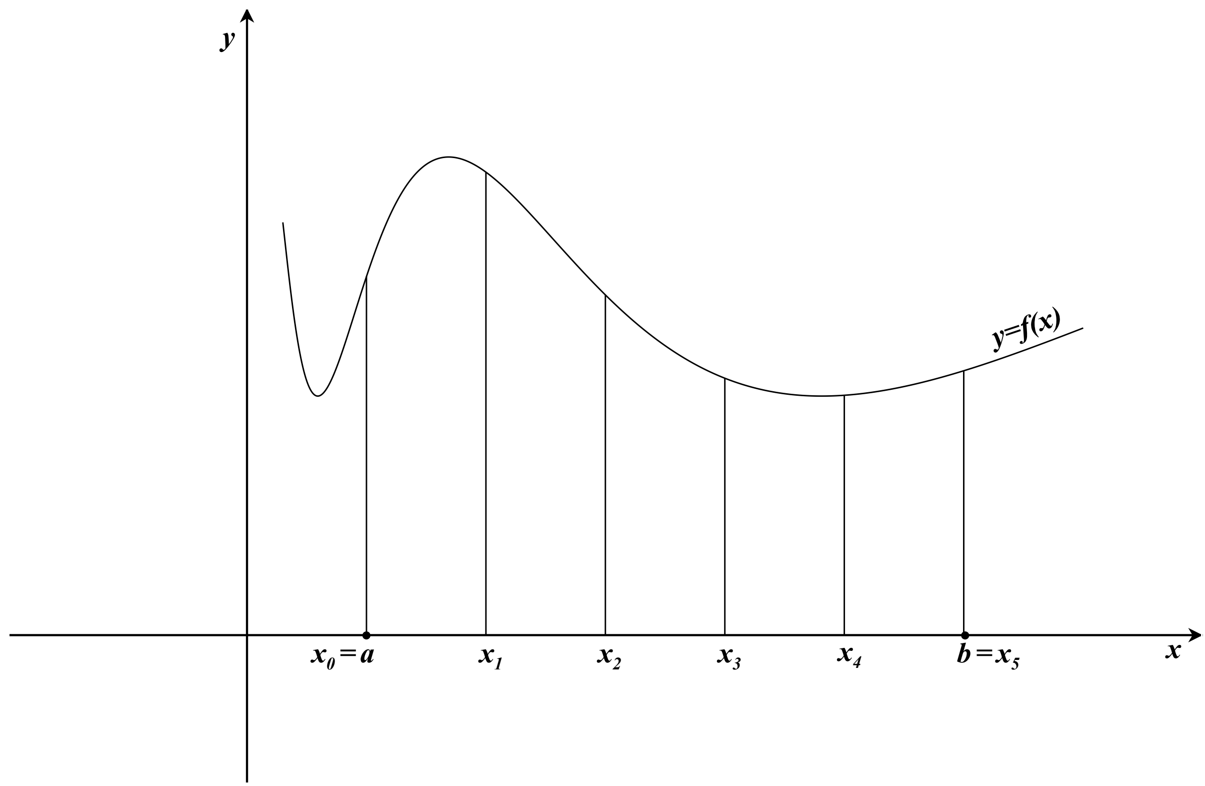
Motivational example:

Find the area of the region R, that is bounded by the lines $y = f(x)$, $x = a$, $x = b$ and the x-axis.



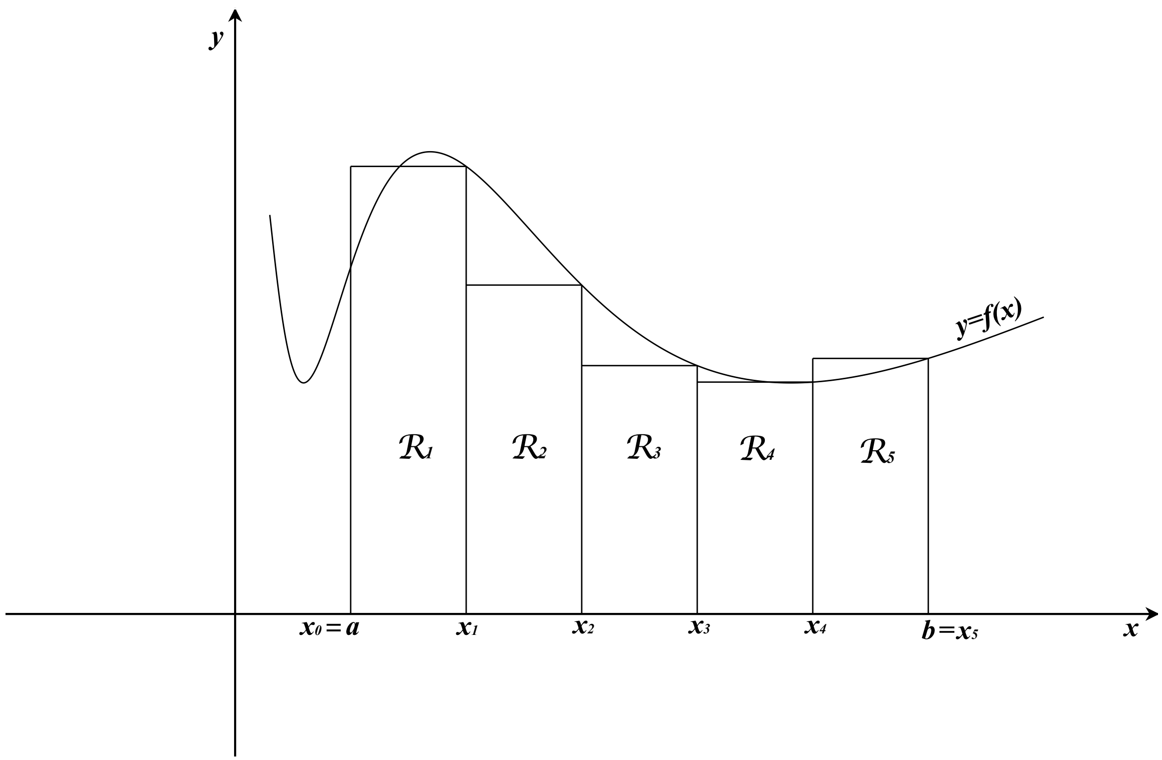
(✻) ***Problem*** — area is defined in terms of *rectangles*, and the region R is not rectangular.

(✻) ***Step 1, Approximation:*** Cover the region R with rectangles, and use the ***sum of the areas*** of these rectangles to ***approximate*** the area of R. To begin, we divide the region into *rectangular strips* ...



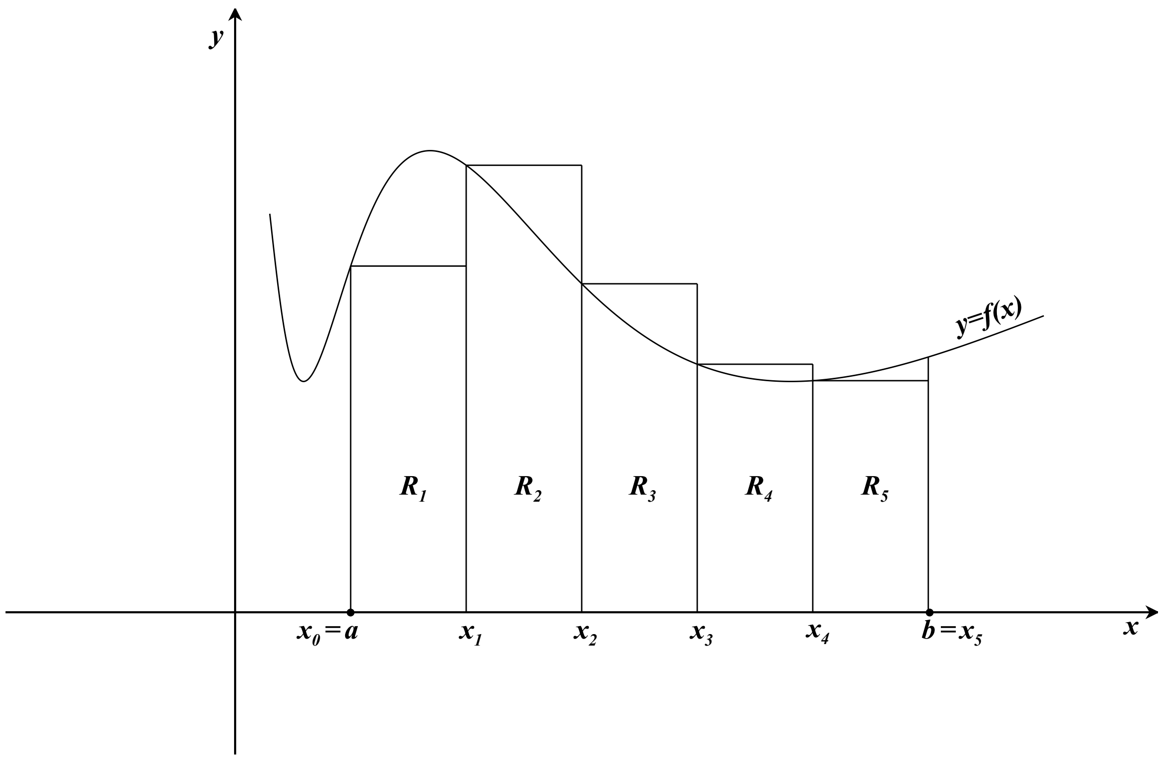
(✻) ... then approximate each strip by a rectangle, using a point on the graph of $y = f(x)$ to determine the height of each rectangle. This can be done in different ways. In figure 1, the height of $\mathcal{R}_j$ is $f(x_j)$ (corresponding to the righthand endpoint of the base of $\mathcal{R}_j$).

Figure 1: Using righthand endpoints to determine heights.



In figure 2, the height of R_j is $f(x_{j-1})$ (corresponding to the lefthand endpoint of the base of R_j).

Figure 2: Using lefthand endpoints to determine heights.



Next, using the second set of rectangles for example, we conclude that

$$\begin{aligned} \text{area}(R) &\approx \sum_{j=1}^{5} \text{area}(R_j) \\ &= \sum_{j=1}^{5} f(x_{j-1}) \cdot (x_j - x_{j-1}) \\ &= \sum_{j=1}^{5} f(x_{j-1}) \cdot \Delta x_j, \end{aligned}$$

where, as usual, $\Delta x_j = (x_j - x_{j-1})$.

(✻) The sum in the third row above is called a ***lefthand sum***, because we used the lefthand endpoints to determine the heights of the rectangles.

(✻) If we use the rectangles in figure 1, we obtain a ***righthand sum*** approximation

$$\text{area}(R) \approx \sum_{j=1}^{5} \text{area}(\mathcal{R}_j) = \sum_{j=1}^{5} f(x_j) \cdot \Delta x_j.$$

(✻) ***Step 2, Refine:*** To *improve* the approximate answer from Step 1, replace the original collection of rectangles by a bigger collection of rectangles, all of which are ***narrower*** than before, as in Figure 3.

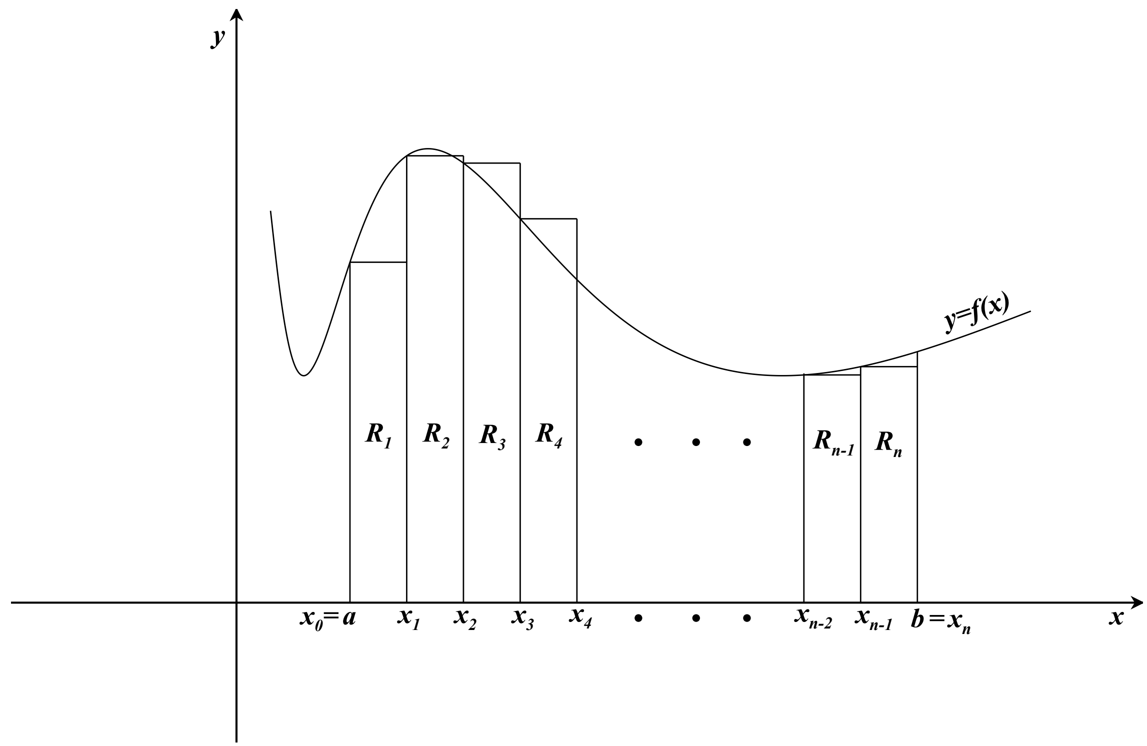


Figure 3: Refining the cover.

(✻) This is called '*refining the cover*'.

Using the rectangles in figure 3 gives the approximation

$$\boxed{\text{area}(R) \approx \sum_{j=1}^{n} \text{area}(R_j) = \sum_{j=1}^{n} f(x_{j-1}) \cdot \Delta x_j}$$

(another *lefthand sum*).

(✻) The refined approximation is typically more accurate than the previous approximation because the narrower rectangles cover the region more accurately.

What next?

Repeat... ***repeat...*** ***repeat...*** ***repeat...***

(✻) ***Step 3, take a limit:*** Continue to refine the collection of rectangles, making the corresponding approximations more and more accurate.

This leads to the conclusion that

$$\boxed{\text{area}(R) = \lim_{n\to\infty}\left(\sum_{j=1}^{n} f(x_{j-1})\cdot\Delta x_j\right),}$$

where

- $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$;
- $\Delta x_j = x_j - x_{j-1}$, for $j = 1, 2, \ldots, n$;

and it is understood that...

as we increase the number of rectangles, the widths of all the rectangles goes to 0:

$$\boldsymbol{\lim_{n\to\infty}\big[\max(\Delta x_j : 1 \le j \le n)\big] = 0}$$

(✻) **Comment:** We don't have to use lefthand sums (or righthand sums).

Definition: The ***definite integral*** of the function $y = f(x)$ on the interval $[a, b]$ is denoted by

$$\int_a^b f(x)\,dx$$

and is defined by the limit

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\sum_{j=1}^{n} f(x_j^*)\cdot\Delta x_j\right),} \tag{1}$$

where for each n:

- $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$;
- x_j^* is *some point* in the interval $[x_{j-1}, x_j]$, i.e., $x_{j-1} \le x_j^* \le x_j$;
- $\Delta x_j = x_j - x_{j-1}$, for $j = 1, 2, \ldots, n$.
- $\lim_{n\to\infty}\big[\max(\Delta x_j : 1 \le j \le n)\big] = 0$.

(✻) The collection of subintervals $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$ is called a *partition* of the original interval $[a, b]$. The length of longest subinterval in any given partition, $\max(\Delta x_j : 1 \le j \le n)$, is called the *diameter* of the partition.