Definition: The ***definite integral*** of the function $y = f(x)$ on the interval $[a, b]$ is denoted by

$$\int_a^b f(x)\,dx$$

and is defined by the limit

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \left(\sum_{j=1}^{n} f(x_j^*) \cdot \Delta x_j \right),$$

where for each n:

- $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$;
- x_j^* is *some point* in the interval $[x_{j-1}, x_j]$, i.e., $x_{j-1} \le x_j^* \le x_j$;
- $\Delta x_j = x_j - x_{j-1}$, for $j = 1, 2, \ldots, n$.
- $\displaystyle\lim_{n\to\infty} \big[\max(\Delta x_j : 1 \le j \le n) \big] = 0.$

(✻) The collection of subintervals $[x_0, x_1]$, $[x_1, x_2]$, …, $[x_{n-1}, x_n]$ is called a *partition* of the original interval $[a, b]$. The length of longest subinterval in any given partition, $\max(\Delta x_j : 1 \le j \le n)$, is called the *diameter* of the partition.

Comments:

(✻) A definite integral $\int_a^b f(x)\,dx$ returns a ***numerical value***.

(✻) If the function $f(x)$ is *continuous* in the interval $[a, b]$, then the limit defining the definite integral *always exists*.

(✻) The value of the limit ***does not depend*** on how the *partitions* are chosen or how the points x_j^* are selected from each subinterval in each subinterval of the partition, as long as the *diameter* of the partition is approaching 0.

(✻) Computing area is an *application* of definite integrals — *not the way that they are defined.*

(✻) If $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ is a partition of $[a, b]$ and $x_{j-1} \leq x_j^* \leq x_j$ for each j, then the sum

$$\sum_{j=1}^{n} f(x_j^*) \cdot \Delta x_j$$

is called a *Riemann sum.*

The most common choice:

(✻) Divide the interval $[a, b]$ into n ***equal*** pieces. Then

$$\Delta x_j = \frac{b-a}{n}$$

and

$$x_j = a + j \cdot \Delta x_j = a + j \cdot \frac{b-a}{n}$$

for each j.

(✻) This also guarantees that $\Delta x_j \to 0$ as $n \to \infty$.

(✻) If $x_j^* = x_{j-1}$, then the resulting (Riemann) sum is a Left hand sum:

$$LHS(n) = \sum_{j=1}^{n} f(x_{j-1}) \Delta x.$$

(✻) If we set $x_j^* = x_j$, then the resulting (Riemann) sum is a Right hand sum:

$$RHS(n) = \sum_{j=1}^{n} f(x_j) \Delta x.$$

Example 1: Find the area of the region bounded by the curve $y = x^2$ and the lines $y = 0$, $x = 1$ and $x = 3$.

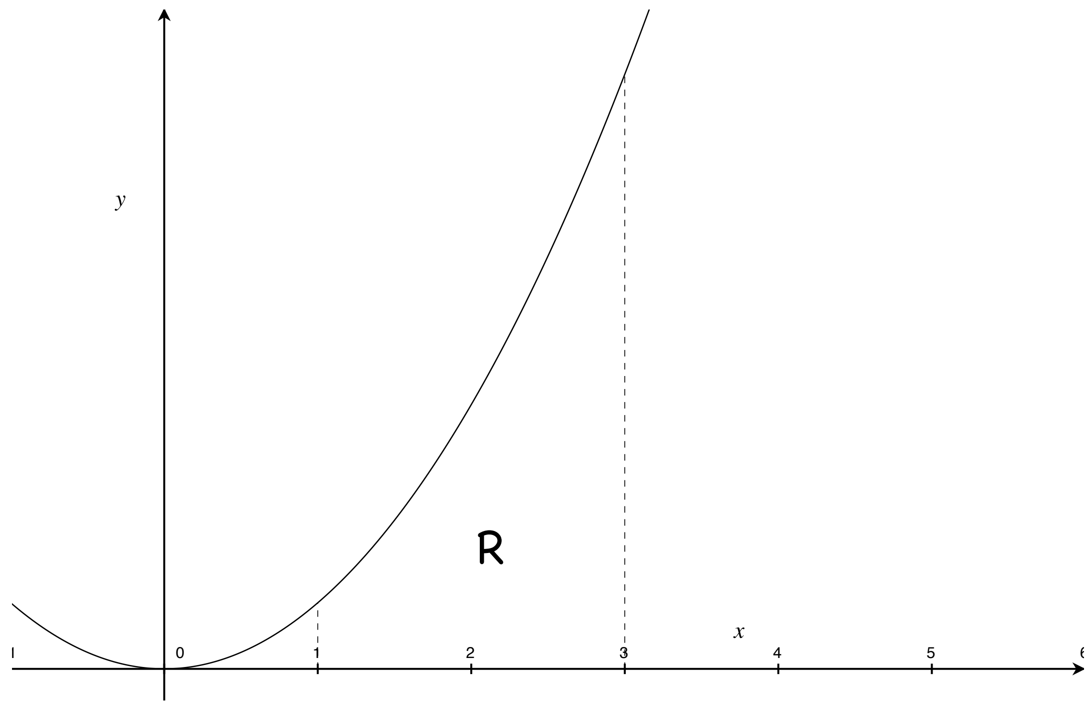


Figure 4: The region in Example 1.

(✱) Use *right hand sums* to calculate

$$\text{area}(\mathcal{R}) = \int_1^3 x^2\, dx.$$

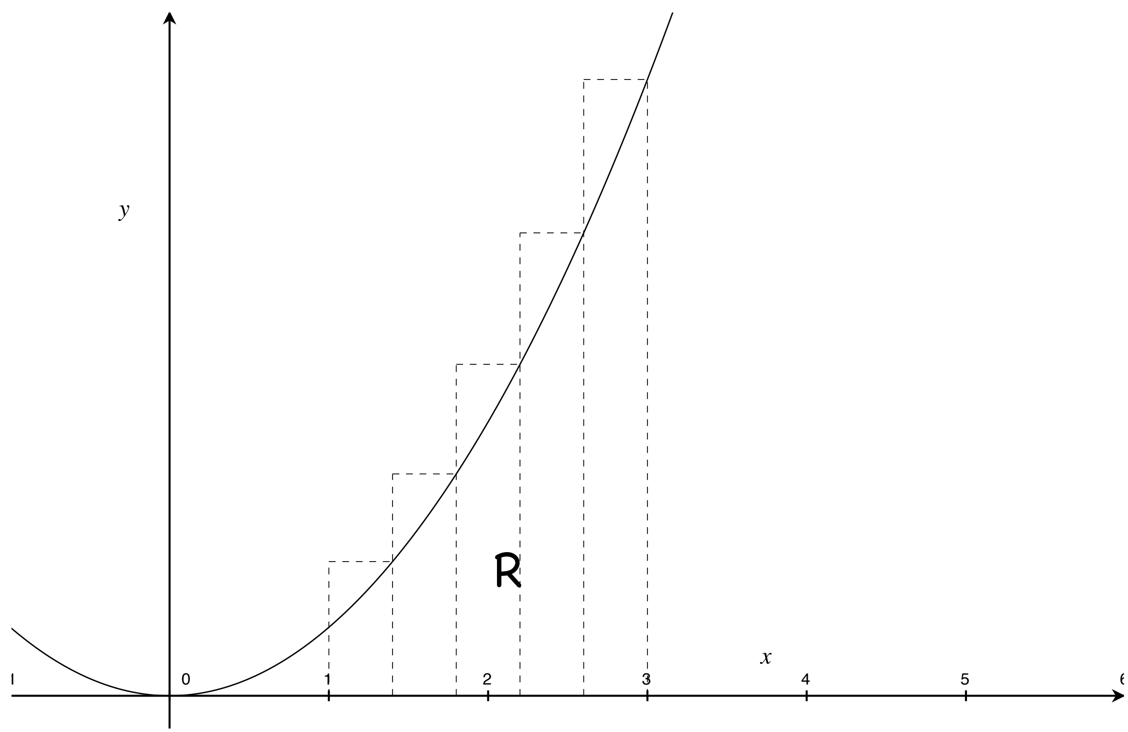


Figure 4: Right hand rectangles.

(✻) Divide the interval $[1, 3]$ into n equal pieces, then...

1. $\Delta x_j = \dfrac{3-1}{n} = \dfrac{2}{n}$, and

2. $x_j = 1 + j \cdot \dfrac{2}{n} = 1 + \dfrac{2j}{n}$.

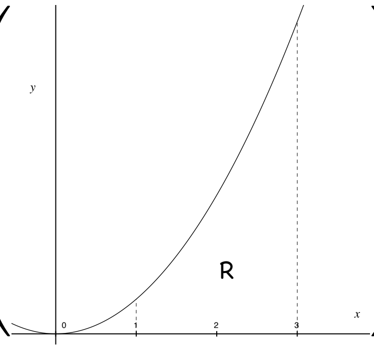
(✱) Write down a right hand sum for this problem:

$$RHS(n) = \sum_{j=1}^{n} f(x_j)\Delta x_j = \sum_{j=1}^{n} x_j^2 \cdot \frac{2}{n} = \sum_{j=1}^{n} \left(1 + \frac{2j}{n}\right)^2 \cdot \frac{2}{n}.$$

(✱) Simplify:

$$\begin{aligned}
RHS(n) &= \sum_{j=1}^{n} \left(1 + \frac{2j}{n}\right)^2 \cdot \frac{2}{n} = \frac{2}{n} \sum_{j=1}^{n} \left(1 + \frac{4j}{n} + \frac{4j^2}{n^2}\right) \\
&= \frac{2}{n} \left(\sum_{j=1}^{n} 1 + \frac{4}{n} \sum_{j=1}^{n} j + \frac{4}{n^2} \sum_{j=1}^{n} j^2 \right) \\
&= \frac{2}{n} \cdot n + \frac{8}{n^2} \cdot \frac{n(n+1)}{2} + \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\
&= 2 + \frac{4n^2 + 4n}{n^2} + \frac{8n^3 + 12n^2 + 4n}{3n^3} \\
&= 2 + 4 + \frac{8}{3} + \frac{4}{n} + \frac{4}{n} + \frac{4}{3n^2} \\
&= \frac{26}{3} + \frac{8}{n} + \frac{4}{3n^2}
\end{aligned}$$

(✱) Take the limit:

area $\Bigg($ 

$\Bigg)$ $= \int_1^3 x^2\,dx$

$$= \lim_{n\to\infty} \left(\sum_{j=1}^{n} \left(1 + \frac{2j}{n}\right)^2 \cdot \frac{2}{n} \right)$$

$$= \lim_{n\to\infty} \left(\frac{26}{3} + \frac{8}{n} + \frac{4}{3n^2} \right)$$

$$= \frac{26}{3}$$

Example 2. Find the area of the region bounded by the lines $y = \frac{1}{2}x+1$, $y = 0$, $x = 0$ and $x = 4$.

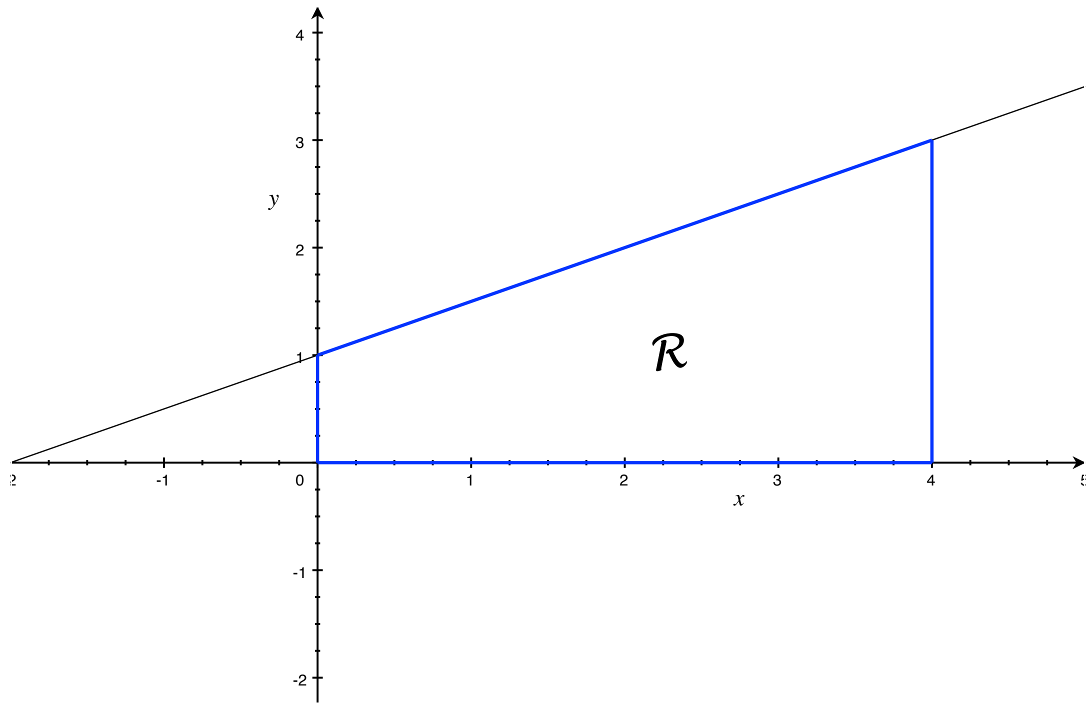
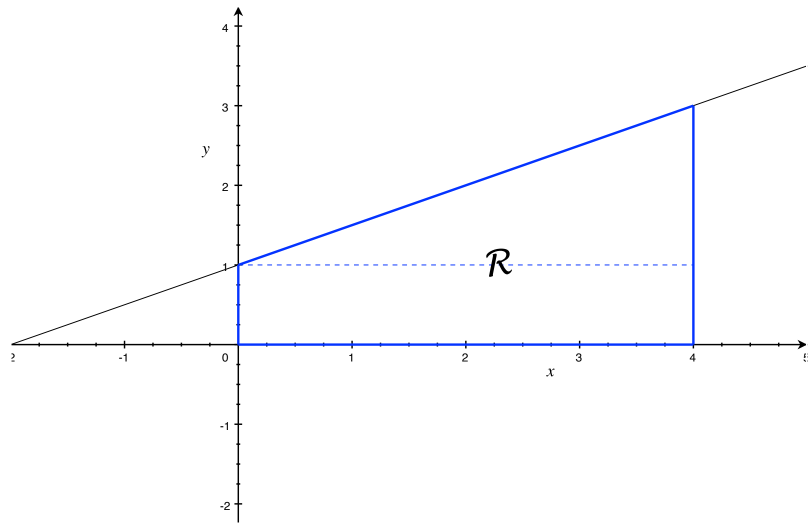


Figure 5: The region in Example 2.

Two methods:

(1) Use formulas from basic geometry to calculate area($\mathcal{R}$)...

(a) Divide $\mathcal{R}$ into a rectangle and a triangle:



(b) Calculate area of triangle and area of rectangle and add:

$$\text{area}(\mathcal{R}) = \text{area(triangle)}+\text{area(rectangle)}$$

$$= 1 \cdot 4 + \frac{(3-1)\cdot 4}{2} = 8.$$

(2) Use a definite integral...

(a) $\text{area}(\mathcal{R}) = \displaystyle\int_0^4 \tfrac{1}{2}x + 1\,dx$

(b) Calculate the integral using righthand sums:

$$\int_0^4 \tfrac{1}{2}x + 1\,dx = \lim_{n\to\infty} \sum_{j=1}^{n} \left(\frac{1}{2}x_j + 1\right)\Delta x_j$$

In this example $\Delta x = \dfrac{4-0}{n} = \dfrac{4}{n}$ and $x_j = 0 + j\Delta x = \dfrac{4j}{n}$, so

$$\int_0^4 \tfrac{1}{2}x + 1\,dx = \lim_{n\to\infty} \sum_{j=1}^{n} \left(\frac{1}{2}\frac{4j}{n} + 1\right)\frac{4}{n} = \lim_{n\to\infty} \left(\frac{4}{n}\sum_{j=1}^{n}\left(\frac{2j}{n} + 1\right)\right)$$

$$= \lim_{n\to\infty}\left(\frac{4}{n}\sum_{j=1}^{n}\frac{2j}{n} + \frac{4}{n}\sum_{j=1}^{n} 1\right) = \lim_{n\to\infty}\left(\frac{8}{n^2}\sum_{j=1}^{n} j + \frac{4}{n}\cdot n\right)$$

$$= \lim_{n\to\infty}\left(\frac{8}{n^2}\cdot\frac{n(n+1)}{2} + 4\right) = \lim_{n\to\infty}\left(4\cdot\frac{n^2+n}{n^2} + 4\right) = 8.$$