The exponential growth model $P = P_0 e^{rt}$ can be quite accurate in the short run, but not in the long run... Because an exponentially growing population will eventually outstrip its resources. This observation leads to a different model...

Logistic growth: This model accounts for the fact that populations grow in environments that generally have a ***carrying capacity***—a maximum sustainable size for the population growing there.

The logistic model is based on the following assumptions/requirements.

(i) When the population is small (relative to the carrying capacity), it should grow at a rate (approximately) proportional to its size (like exponential growth).

(ii) As the population approaches the carrying capacity, the growth rate should approach 0.

(iii) If the initial population size is bigger than the carrying capacity, then the growth rate should be negative.

(iv) The model should be as simple as possible.

If the carrying capacity is M and the intrinsic growth rate is r, then the first three assumptions translate to

(i) If $P/M \approx 0$, then $\frac{dP}{dt} \approx rP$.

(ii) If $P/M \approx 1$, then $\frac{dP}{dt} \approx 0$.

(iii) If $P/M > 1$, then $\frac{dP}{dt} < 0$.

These assumptions (and the desire for as simple a model as possible), lead to the ***logistic equation***:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right),$$

which satisfies all three conditions:

(*) If $P/M \approx 0$, then $rP\left(1 - \frac{P}{M}\right) \approx rP(1-0) = rP$

(*) If $P/M \approx 1$, then $rP\left(1 - \frac{P}{M}\right) \approx rP(1-1) = 0$

(*) If $P/M > 1$, then $rP\left(1 - \frac{P}{M}\right) < 0$

The logistic equation is separable and is solved as follows.

First, factor out $1/M$ from the second factor on the right

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right) = \frac{r}{M}P\left(M - P\right).$$

Then separate

$$\frac{dP}{P(M-P)} = \frac{r}{M}\,dt.$$

Then integrate (using formula #5 in the appendix, with $a = M$ and $b = -1$)

$$\int \frac{dP}{P(M-P)} = \int \frac{r}{M}\,dt \implies \frac{1}{M}\ln\left|\frac{P}{M-P}\right| = \frac{rt}{M} + C.$$

Finally, solve for P

$$\frac{1}{\cancel{M}}\ln\left|\frac{P}{M-P}\right| = \frac{rt}{\cancel{M}} + C \implies \ln\left|\frac{P}{M-P}\right| = rt + C$$

$$\implies \frac{P}{M-P} = Ae^{rt}$$

where $A = \pm e^C$.

A little more algebra:

$$P = (M - P)Ae^{rt} = AMe^{rt} - APe^{rt} \implies P + APe^{rt} = AMe^{rt}$$

$$\implies P(1 + Ae^{rt}) = AMe^{rt}$$

$$\implies P = \frac{AMe^{rt}}{1 + Ae^{rt}}$$

The formula for $P(t)$ can be further manipulated in different ways.

(*) A common approach is to divide the numerator and denominator by Ae^{rt} which gives

$$P = \frac{M}{1 + be^{-rt}},$$

where $b = A^{-1}$. (Our textbook does it this way.)

(*) Another approach is to replace A by a more meaningful parameter. Both M and r have meaningful interpretations, and it is relatively easy to express A in terms of M and the *initial population size* P_0.

If $t = 0$, then

$$P_0 = P(0) = \frac{AM}{1+A} \implies AM = P_0(1+A) = P_0 + AP_0$$

$$\implies AM - AP_0 = P_0 \implies A(M - P_0) = P_0$$

$$\implies A = \frac{P_0}{M - P_0}$$

Now, substitute this for A in the first expression for P

$$P = \frac{AMe^{rt}}{1 + Ae^{rt}} \implies \frac{\frac{P_0}{M-P_0} Me^{rt}}{1 + \frac{P_0}{M-P_0} e^{rt}}$$

Finally, multiply both top and bottom by $(M - P_0)e^{-rt}$, which gives

$$P(t) = \frac{P_0 M}{P_0 + (M - P_0)e^{-rt}}$$

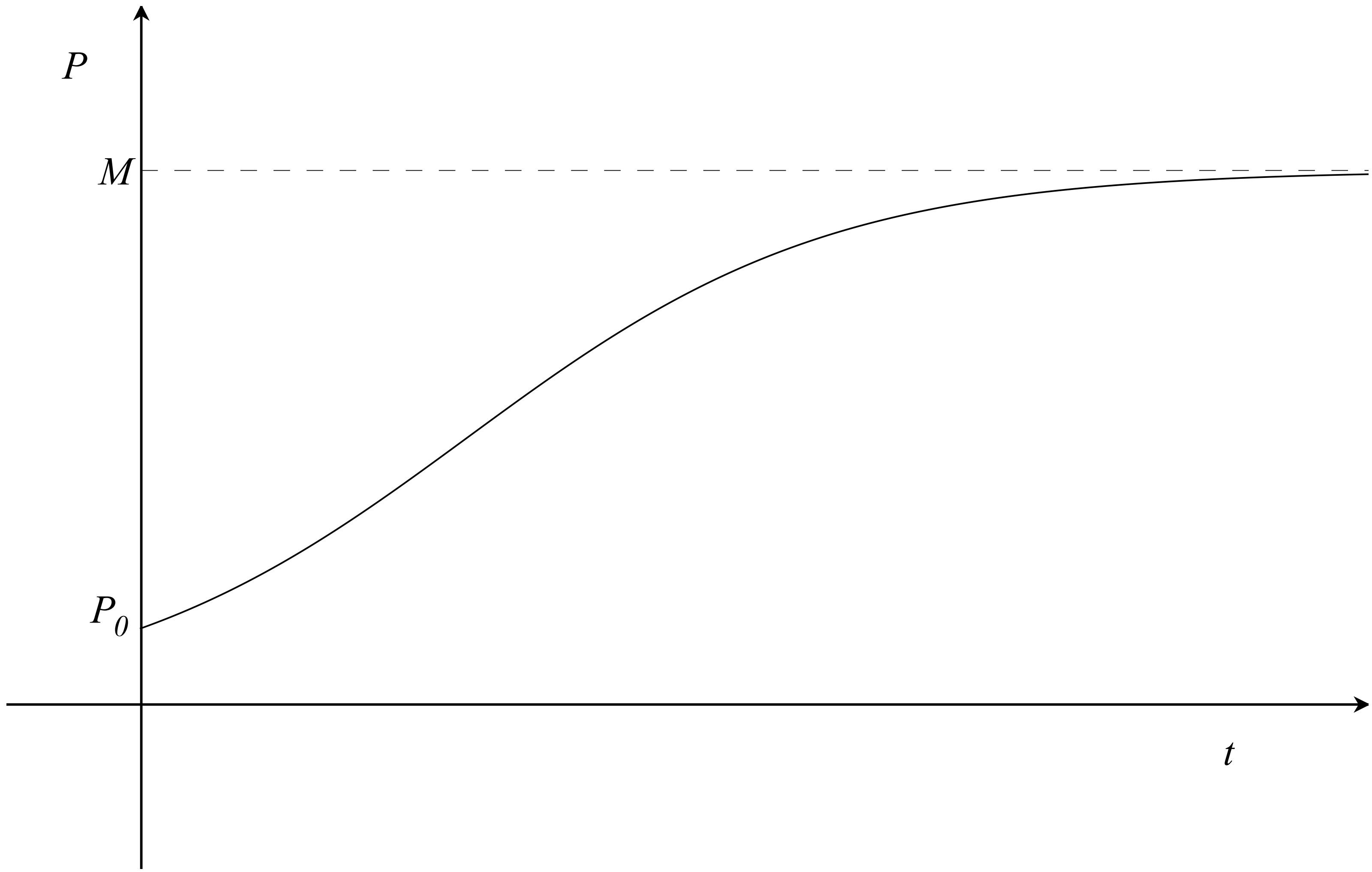

Figure 1: Graph of $P = \dfrac{P_0 M}{P_0 + (M - P_0)e^{-rt}}$

Example. A new virus is spreading on a closed network of 5000 computers. By the time the virus is first spotted, 25 computers are infected, and two hours later 200 computers are infected. Assuming logistic growth, how many hours before half the network is infected?

In this example, we know the carrying capacity $M = 5000$ and the initial population size $P_0 = 25$, so the number of infected computers at time t is

$$P(t) = \frac{25 \cdot 5000}{25 + 4975e^{-rt}} = \frac{5000}{1 + 199e^{-rt}}.$$

From the data, we have

$$P(3) = \frac{5000}{1 + 199e^{-2r}} = 200 \implies 5000 = 200(1 + 199e^{-2r})$$

$$\implies 25 = 1 + 199e^{-2r}$$

$$\implies 24 = 199e^{-2r}$$

$$\implies e^{2r} = \frac{199}{24}$$

$$\implies r = \frac{1}{2}\ln(199/24)$$

Finally, solve the equation $P(t_1) = 5000/2 = 2500$...

$$2500 = \frac{5000}{1 + 199e^{-rt_1}} \implies 1 + 199e^{-rt_1} = \frac{5000}{2500} = 2$$

$$\implies 199e^{-rt_1} = 1 \implies e^{-rt_1} = \frac{1}{199} \implies e^{rt_1} = 199$$

$$\implies t_1 = \frac{\ln 199}{r} = \frac{\ln 199}{\frac{1}{2}\ln(199/24)} \approx 5$$

Conclusion: Half the network will be infected about 5 hours after the virus is first detected.

Differential calculus in several variables.

Example. A demand function

$$q = 5\sqrt{8y + 7p_s - 4p},$$

where

- q is the monthly demand for a firm's good, measured in 1000s of units.
- y is the average monthly income in the market for the firm's good, measured in 1000s of dollars.
- p_s is the average price of substitutes for a firm's good, measured in dollars.
- p is the price of the firm's good, measured in dollars.

(*) If the current prices are $p = \$10$ and $p_s = \$9$, and the average income is \$3300, then the demand will be

$$q\Big|_{\substack{y=3.3\\ ps=9\\ p=10}} = 5\sqrt{26.4 + 63 - 40} \approx 35.143 \implies \approx 35{,}143 \text{ units}$$

Question: What will happen to demand if the prices stay the same, but average income decreases to $y_1 = 3000$?

Qualitative answer: Demand will decrease.

Quantitative answer:

$$q\Big|_{\substack{y=3\\ ps=9\\ p=10}} = 5\sqrt{24+63-40} \approx 34.278 \implies \approx 34,278 \text{ units}$$

I.e., demand will decrease by about $35,143 - 34,278 = 865$ units.

(*) Key assumption: when calculating the effect of the change in income on the demand, we ***hold the other variables (the prices) fixed***.

(*) We used derivatives to analyze the behavior of functions of one variable, for example in optimization problems. We want to be able to do the same when there are more variables.

Definition: If $w = f(x, y, z)$, then the partial derivative of w with respect to x is defined by

$$\frac{\partial w}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

Likewise, the partial derivatives of w with respect to y and z are

$$\frac{\partial w}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

and

$$\frac{\partial w}{\partial z} = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

Comments:

(i) We use the *'curly d'* symbol (∂) to distinguish partial derivatives from ordinary derivatives (of functions of one variable).

(ii) When differentiating with respect to one variable, the other variables are ***held fixed***.

(iii) The usual rules of differentiation still hold... Yay!

Example. Returning to the demand function

$$q = 5\sqrt{8y + 7p_s - 4p} = 5(8y + 7p_s - 4p)^{1/2},$$

The partial derivative of q with respect to income is

$$\frac{\partial q}{\partial y} = 5 \cdot \frac{1}{2}(8y + 7p_s - 4p)^{-1/2} \cdot 8 = \frac{20}{(8y + 7p_s - 4p)^{1/2}},$$

(*) I used the rule for powers and the chain rule, and most importantly, I treated p and p_s as constants.

I.e., when differentiating with respect to y, we can think of the demand function as being

$$q = 5(8y + C)^{1/2}$$

where $C = 7p_s - 4p$. Similarly, the other partial derivatives are

$$\frac{\partial q}{\partial p_s} = 5 \cdot \frac{1}{2}(8y + 7p_s - 4p)^{-1/2} \cdot 7 = \frac{35}{2(8y + 7p_s - 4p)^{1/2}}$$

and

$$\frac{\partial q}{\partial p} = 5 \cdot \frac{1}{2}(8y + 7p_s - 4p)^{-1/2} \cdot (-4) = -\frac{10}{(8y + 7p_s - 4p)^{1/2}}$$