Techniques of integration: Substitution

Observation: If $u = g(x)$, then $du = g'(x)\,dx$.

We can use this observation to simplify integrals that have the form

$$\int f(g(x))\,g'(x)\,dx,$$

by ***substituting*** $u = g(x)$, which entails $du = g'(x)\,dx$:

$$\int f(\overbrace{g(x)}^{u})\,\overbrace{g'(x)\,dx}^{du} = \int f(u)\,du$$

Example 1: The substitution $u = x^2 + 3$, $du = 2x\,dx$ gives

$$\begin{aligned}\int 2x\sqrt{x^2+3}\,dx &= \int (\overbrace{x^2+3}^{u})^{1/2}\,\overbrace{2x\,dx}^{du} \\ &= \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{3}(x^2+3)^{3/2} + C\end{aligned}$$

(*) The key is *recognizing* that a substitution is possible.

(*) What we are looking for in the integrand is a *product*, with one factor being a *composite function*, e.g., $f(g(x))$, and the other factor being the *derivative of the argument* of the first factor, i.e., $g'(x)$.

Example 2: Compute

$$\int \frac{2x}{x^2+4}\,dx$$

The integrand looks like a quotient, but really it's a product, and one of the factors is composite:

$$\int \frac{2x}{x^2+4}\,dx = \int (x^2+4)^{-1}\,2x\,dx$$

Substituting $u = x^2+4$ and $du = 2x\,dx$ makes the integral easy to compute:

$$\int (\overbrace{x^2+4}^{u})^{-1}\,\overbrace{2x\,dx}^{du} = \int u^{-1}\,du = \ln|u| + C = \ln(x^2+4) + C.$$

Observation: We can multiply or divide any equation by a *nonzero* constant. This makes substitution more flexible.

Example 3: Compute

$$\int 5x^2 \sqrt[4]{2x^3+1}\,dx$$

The natural choice for a substitution is $u = 2x^3 + 1$, which entails $du = 6x^2\,dx$. Dividing this equation by 6, shows that

$$x^2\,dx = \frac{1}{6}\,du \implies 5x^2\,dx = \frac{5}{6}\,du,$$

so the substitution $u = 2x^3 + 1$ will work here:

$$\begin{aligned}\int 5x^2 \sqrt[4]{2x^3+1}\,dx &= \int (\overbrace{2x^3+1}^{u})^{1/4}\,\overbrace{5x^2\,dx}^{\frac{5}{6}\,du} \\ &= \frac{5}{6}\int u^{1/4}\,du = \frac{2}{3}u^{5/4} + C \\ &= \frac{2}{3}(2x^3+1)^{5/4} + C\end{aligned}$$

Observation: If $u = ax + b$, then $du = a\,dx$ so that $dx = \dfrac{1}{a}\,du$. This means that we can always make the following simplification

$$\int f(ax+b)\,dx = \frac{1}{a}\int f(u)\,du$$

by substituting $u = ax + b$.

Example 4: Compute

$$\int \sqrt[3]{5x+2}\,dx = \int (5x+2)^{1/3}\,dx.$$

The substitution $u = 5x + 2, du = 5\,dx \implies dx = \frac{1}{5}\,du$ will work here:

$$\int (\overbrace{5x+2}^{u})^{1/3}\overbrace{dx}^{\frac{1}{5}\,du} = \int u^{1/3}\cdot\frac{1}{5}\,du = \frac{3}{20}u^{4/3} + C = \frac{3}{20}(5x+2)^{4/3} + C$$

Example 5: Find the demand equation for a firm whose marginal revenue function is

$$\frac{dr}{dq} = 10\sqrt[3]{1000 - 0.5q}.$$

Observation: Since $r = pq$, by definition, we can find the demand equation for this firm by

(i) Finding the revenue function $r(q)$, and

(ii) Writing $p = r(q)/q$.

Step 1. Integrate, by substituting $u = 1000 - 0.5q$, $du = -0.5\,dq \implies dq = -2\,du$:

$$\begin{aligned}\int 10\sqrt[3]{1000 - 0.5q}\,dq &= 10\int (\overbrace{1000 - 0.5q}^{u})^{1/3}\,\overbrace{dq}^{-2\,du} \\ &= -20\int u^{1/3}\,du \\ &= -15u^{4/3} + C = -15(1000 - 0.5q)^{4/3} + C\end{aligned}$$

Conclusion 1: $r = C - 15(1000 - 0.5q)^{4/3}$

Step 2: Solve for C:

$$0 = r(0) = C - 15(1000)^{4/3} \implies C = 150000.$$

Conclusion: $r = 150000 - 15(1000 - 0.5q)^{4/3}$.

Step 3: Write down the demand equation:

$$p = \frac{r}{q} = \frac{15\left(10000 - (1000 - 0.5q)^{4/3}\right)}{q}$$

Observation: Linear equations are easy to solve. For example, if $u = ax + b$, then

$$x = \frac{1}{a}(u - b).$$

Example 6: Compute

$$\int \frac{2x}{3x+4}\,dx.$$

The substitution $u = 3x+4$, $du = 3\,dx \implies dx = \frac{1}{3}\,du$ can also be used to express $2x$ in terms of u: $x = \frac{1}{3}(u-4) \implies 2x = \frac{2}{3}(u-4)$, so

$$\begin{aligned}
\int \frac{\overbrace{2x}^{\frac{2}{3}(u-4)}}{\underbrace{3x+4}_{u}} \overbrace{dx}^{\frac{1}{3}\,du} &= \int \frac{\frac{2}{3}(u-4)}{u}\,\frac{1}{3}du \\
&= \frac{2}{9}\int 1 - \frac{4}{u}\,du \\
&= \frac{2}{9}u - \frac{8}{9}\ln|u| + C \\
&= \frac{2}{9}(3x+4) - \frac{8}{9}\ln|3x+4| + C \\
&= \frac{2}{3}x - \frac{8}{9}\ln|3x+4| + C
\end{aligned}$$