The chain rule (in several variables)

Suppose that $w = f(x, y, z)$ and $x = x(\alpha, \beta)$ and $y = y(\alpha, \beta)$...

then...

$$\frac{\partial w}{\partial \alpha} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial \alpha} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial \alpha} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial \alpha}$$

and

$$\frac{\partial w}{\partial \beta} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial \beta} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial \beta} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial \beta}$$

Explanation... Use ***linear approximation:***

Suppose that α changes by $\Delta\alpha \approx 0$ (and β remains fixed). Then

$$\Delta x \approx \frac{\partial x}{\partial \alpha}\Delta\alpha, \quad \Delta y \approx \frac{\partial y}{\partial \alpha}\Delta\alpha \quad \text{and} \quad \Delta z \approx \frac{\partial z}{\partial \alpha}\Delta\alpha$$

Now, if x, y and z all change, then w changes by...

$$\begin{aligned}\Delta w &\approx \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \frac{\partial w}{\partial z}\Delta z \\ &\approx \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial \alpha}\Delta\alpha\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial \alpha}\Delta\alpha\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial \alpha}\Delta\alpha\right) \\ &= \left(\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \alpha} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \alpha} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial \alpha}\right)\Delta\alpha \ldots\end{aligned}$$

Therefore $\dfrac{\Delta w}{\Delta\alpha} \approx \dfrac{\partial w}{\partial x}\cdot\dfrac{\partial x}{\partial \alpha} + \dfrac{\partial w}{\partial y}\cdot\dfrac{\partial y}{\partial \alpha} + \dfrac{\partial w}{\partial z}\cdot\dfrac{\partial z}{\partial \alpha}$,

so

$$\frac{\partial w}{\partial \alpha} = \lim_{\Delta\alpha\to 0}\frac{\Delta w}{\Delta\alpha} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \alpha} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \alpha} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial \alpha}$$

Profit-maximizing: a simple example

Suppose that a firm's profit function is

$$\Pi = f(p) = -0.4p^2 + 60p - 500,$$

where p is the price that the firm sets and Π is the firm's weekly profit.

To find the price that maximizes profit and the maximum weekly profit, we proceed as usual:

(✻) $d\Pi/dp = 0 \implies -0.8p + 60 = 0 \implies p^* = 60/0.8 = 75$

(✻) $d^2\Pi/dp^2 = -0.8 < 0 \implies \Pi^* = \Pi(p^*) = 1750$ is the maximum profit and $p^* = 75$ is the profit maximizing price.

(✻) **Question:** what will happen to the firm's maximum weekly profit if the ***parameter*** $\alpha = 0.4$ changes to $\alpha_1 = 0.42$?

Variables and parameters

Generalizing the profit maximizing example:

$$\Pi = f(p; \alpha, \beta, \gamma) = -\alpha p^2 + \beta p - \gamma,$$

where (as before)

(✻) p is the price that the firm sets, and

(✻) α, β and γ are quantities that are determined by the market in which the firm operates.

(✻) The price p is called an ***endogenous*** variable, or just a variable, because its value is set *within* the model.

(✻) The quantities α, β and γ are called ***exogenous*** variables, or ***parameters***, because their values are set *outside* the model.

(✻) *When the firm is working with its profit model — e.g., to find the price which maximizes profit — the parameters α, β and γ are treated as constants.*

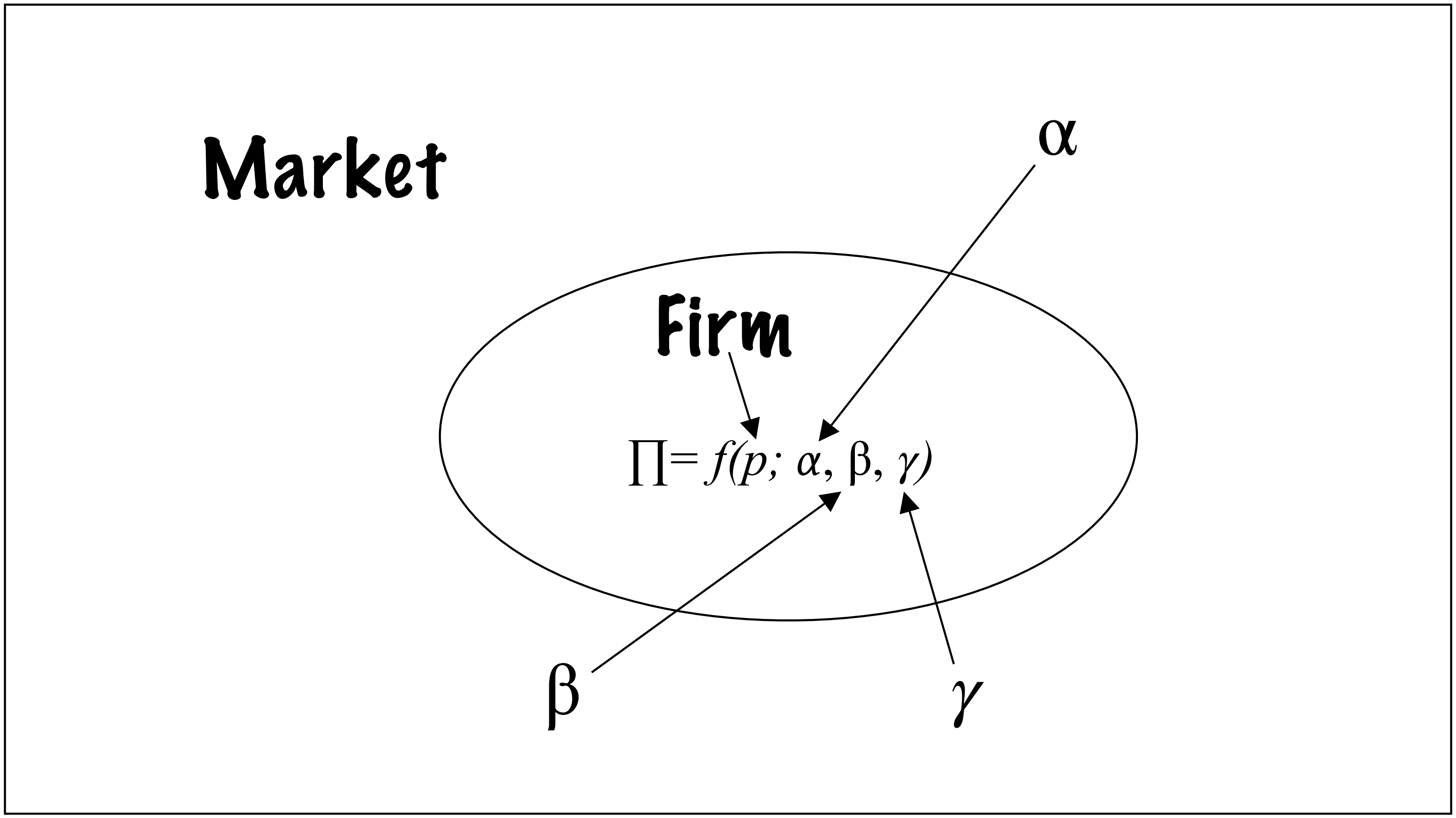

Figure 1: The firm's profit model, in the bigger model of the market.

To maximize its profit...

(✻) The firm finds the critical price p^* by solving the first order equation

$$\frac{d\Pi}{dp} = 0 \implies -2\alpha p + \beta = 0 \implies p^* = \frac{\beta}{2\alpha}.$$

(✻) Finds the critical profit

$$\Pi^* = -\alpha\left(p^*\right)^2 + \beta p^* - \gamma = -\frac{\beta^2}{4\alpha} + \frac{\beta^2}{2\alpha} - \gamma = \frac{\beta^2}{4\alpha} - \gamma$$

(✻) Verifies that the critical profit is a maximum

$$\frac{d^2\Pi}{dp^2} = -2\alpha < 0 \implies \Pi^* \text{ is the max profit}$$

if the parameter $\alpha > 0$.

Key observation:

The critical values of the price p and the profit Π are both ***functions*** *of the parameters, α, β and γ:*

$$p^* = \frac{\beta}{2\alpha}$$
$$\Pi^* = \frac{\beta^2}{4\alpha} - \gamma$$

So....

If market forces cause the values of the parameters to change, then the critical price and the critical profit will both change.

Key Question: *How do (small) changes in the parameters affect the critical values? More specifically, what are the rates of change*

$$\frac{\partial \Pi^*}{\partial \alpha}, \quad \frac{\partial \Pi^*}{\partial \beta} \quad \text{and} \quad \frac{\partial \Pi^*}{\partial \gamma}\ ?$$

In this hypothetical example, we can answer this directly...

$$\frac{\partial \Pi^*}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left(\frac{\beta^2}{4\alpha} - \gamma\right) = -\frac{\beta^2}{4\alpha^2}$$

$$\frac{\partial \Pi^*}{\partial \beta} = \frac{\partial}{\partial \beta}\left(\frac{\beta^2}{4\alpha} - \gamma\right) = \frac{\beta}{2\alpha}$$

$$\frac{\partial \Pi^*}{\partial \gamma} = \frac{\partial}{\partial \gamma}\left(\frac{\beta^2}{4\alpha} - \gamma\right) = -1$$

In particular

$$\left.\frac{\partial \Pi^*}{\partial \alpha}\right|_{\substack{\alpha=0.4\\ \beta=60\\ \gamma=500}} = -\frac{3600}{4 \cdot 0.16} = -5625$$

so if $\Delta\alpha = 0.02$ (α changes from 0.4 to 0.42), then

$$\Delta\Pi^* \approx \left.\frac{\partial \Pi^*}{\partial \alpha}\right|_{\substack{\alpha=0.4\\ \beta=60\\ \gamma=500}} \cdot \Delta\alpha = -5625 \cdot 0.02 = -112.5$$

But...

In general, it can be tricky to express the critical values as explicit functions of the parameters.

Happily there is a convenient work-around

based on the chain rule...

The Envelope Theorem

Suppose we have a function $F(x, y, z; \alpha, \beta)$ that depends on three variables, x, y and z, and two parameters, α and β.

And also suppose that (x^*, y^*, z^*) is a critical point of this function. This means that

$$\begin{aligned} F_x(x^*, y^*, z^*; \alpha, \beta) &= 0 \\ F_y(x^*, y^*, z^*; \alpha, \beta) &= 0 \\ F_z(x^*, y^*, z^*; \alpha, \beta) &= 0 \end{aligned}$$

and F^* is the corresponding critical value:

$$F^* = F(x^*, y^*, z^*; \alpha, \beta).$$

Just as in the profit-maximizing example, the critical point and critical value are functions of the parameters:

$$x^* = x^*(\alpha, \beta),\ y^* = y^*(\alpha, \beta),\ z^* = z^*(\alpha, \beta) \quad \text{and} \quad F^* = F^*(\alpha, \beta).$$

To find $\partial F^*/\partial\alpha$ and $\partial F^*/\partial\beta$, we use the chain rule:

$$\begin{aligned}
\frac{\partial F^*}{\partial \alpha} &= \frac{\partial}{\partial \alpha} F(x^*, y^*, z^*; \alpha, \beta) \\
&= \cancelto{0}{\left.\frac{\partial F}{\partial x}\right|_{\substack{x=x^*\\y=y^*\\z=z^*}}} \cdot \frac{\partial x^*}{\partial \alpha} + \cancelto{0}{\left.\frac{\partial F}{\partial y}\right|_{\substack{x=x^*\\y=y^*\\z=z^*}}} \cdot \frac{\partial y^*}{\partial \alpha} + \cancelto{0}{\left.\frac{\partial F}{\partial z}\right|_{\substack{x=x^*\\y=y^*\\z=z^*}}} \cdot \frac{\partial z^*}{\partial \alpha} \\
&\qquad + \left.\frac{\partial F}{\partial \alpha}\right|_{\substack{x=x^*\\y=y^*\\z=z^*}} \cdot \cancelto{1}{\frac{\partial \alpha}{\partial \alpha}} + \left.\frac{\partial F}{\partial \beta}\right|_{\substack{x=x^*\\y=y^*\\z=z^*}} \cdot \cancelto{0}{\frac{\partial \beta}{\partial \alpha}} \\
&= F_\alpha(x^*, y^*, z^*; \alpha, \beta)
\end{aligned}$$

(Assuming that α and β are independent of each other!)

In words: *The partial derivative of the critical value F^* with respect to a parameter α is equal to the partial derivative of the original function F with respect to α, evaluated at the critical point.*

Returning to the profit-maximizing example (in which case there is one variable and three parameters)...

The original profit function was

$$\Pi = -\alpha p^2 + \beta p - \gamma$$

and the critical point was

$$p^* = \frac{\beta}{2\alpha}.$$

Now,

$$\Pi_\alpha = -p^2,$$

so according to the *envelope theorem*

$$\frac{\partial \Pi^*}{\partial \alpha} = \Pi_\alpha(p^*; \alpha, \beta, \gamma) = -(p^*)^2 = -\frac{\beta^2}{4\alpha^2}$$

which agrees with what we found before.