Observation: This situation is analogous to what we saw in the beginning of the quarter with indefinite integrals.

If $f(x)$ is continuous, then there are infinitely many solutions to the (differential) equation

$$y' = f(x)$$

given by the indefinite integral

$$y = \int f(x)\,dx = F(x) + C,$$

where $F(x)$ is any (particular) antiderivative of $f(x)$.

But there is only one solution to the initial value problem

$$y' = f(x); \quad y(x_0) = y_0,$$

which we find by using the data $y(x_0) = y_0$ to solve for C

$$y = F(x) + \overbrace{(y_0 - F(x_0))}^{C}.$$

The amount of data that we require to find a specific solution depends on the number of unspecified parameters that appear in the general solution of the differential equation.

Example 2. (cont.) We saw that if labor-elasticity of output ($\eta_{q/l}$) is constant, then the production function must have the form

$$q = Al^{\eta_0},$$

where η_0 is the value of the constant elasticity.

If η_0 is known, then we just need one data point to find A, but if η_0 is unknown, we will need two data points...

Suppose that the production function we seek has constant labor-elasticity and also satisfies $q(20) = 200$ and $q(30) = 250$. This data leads to a pair of equations for the unknown parameters A and η_0, which we solve as follows:

$$\left.\begin{array}{rcl} A \cdot 20^{\eta_0} & = & 200 \\ A \cdot 30^{\eta_0} & = & 250 \end{array}\right\} \Longrightarrow \frac{\cancel{A}30^{\eta_0}}{\cancel{A}20^{\eta_0}} = \frac{250}{200} \Longrightarrow \left(\frac{3}{2}\right)^{\eta_0} = \frac{5}{4}.$$

Taking logarithms of both sides of the right-most equation gives

$$\eta_0 \ln(3/2) = \ln(5/4) \Longrightarrow \eta_0 = \frac{\ln(5/4)}{\ln(3/2)} \quad (\approx 0.55034),$$

and

$$A = 200 \cdot 20^{-\eta_0} \approx 38.461.$$

I.e.,

$$q \approx 38.461 \cdot l^{0.55034}$$

is the production function we seek.

Summary: To solve the (separable) 'initial' value problem

$$\Phi(y, y', x) = 0; \quad y(x_0) = y_0,\, y(x_1) = y_1, \ldots, y(x_n) = y_n$$

(i) Separate:

$$\Phi(y, y', x) = 0 \implies h(y)\frac{dy}{dx} = g(x) \implies h(y)\, dy = g(x)\, dx.$$

(ii) Integrate (if possible):

$$h(y)\, dy = g(x)\, dx \implies \int h(y)\, dy = \int g(x)\, dx \implies H(y) = G(x) + C.$$

(iii) Solve for y (if possible):

$$H(y) = G(x) + C \implies y = H^{-1}\big(G(x) + C\big)\,.$$

(iv) Use data to solve for C (and any other unspecified parameters in H and G). The number of data points should generally match the number of unknown constants.

Example 3. Solve the initial value problem

$$y' = \frac{3x+1}{y}; \quad y(1) = 2.$$

Separate: $\quad y' = \dfrac{3x+1}{y} \implies y\dfrac{dy}{dx} = 3x+1 \implies y\,dy = (3x+1)\,dx$

Integrate: $\quad \displaystyle\int y\,dy = \int 3x+1\,dx \implies \frac{1}{2}y^2 = \frac{3}{2}x^2 + x + C.$

Solve for y:

$$\frac{1}{2}y^2 = \frac{3}{2}x^2 + x + C \implies y^2 = 3x^2 + 2x + C \implies y = \pm\sqrt{3x^2 + 2x + C}$$

Solve for C (and the $\pm$): Since $y(1) = 2 > 0$, we choose the $+$ sign. As for C:

$$y(1) = 2 \implies 2 = +\sqrt{3 \cdot 1^2 + 2 \cdot 1 + C} = \sqrt{5 + C} \implies C = -1,$$

so the solution is

$$y = \sqrt{3x^2 + 2x - 1}.$$

Example 4. A metal ball is heated to a temperature of 100°C and then dropped into a tank of water kept at a constant temperature of 5°C. After 20 minutes, the temperature of the ball is measured to be 73°C. When will the temperature of the ball reach a temperature of 20°C?

Recall that *Newton's law of cooling* says that

$$\frac{dT}{dt} = k(T - \tau)$$

where

- $T = T(t)$ is the temperature the body, as a function of time t.
- τ is the constant ambient temperature of the medium (water in this case).
- k is an unknown constant that depends on the physical characteristics of the body and the medium.

In this example, $\tau = 5^\circ$, so the differential equation we need to solve here is

$$\frac{dT}{dt} = k(T - 5).$$

Separate: $\dfrac{dT}{dt} = k(T-5) \implies \dfrac{dT}{T-5} = k\,dt.$

Integrate: $\displaystyle\int \frac{dT}{T-5} = \int k\,dt \implies \ln|T-5| = kt + C.$

Solve for T:

$$
\begin{aligned}
e^{\ln|T-5|} = e^{kt+C} &\implies |T-5| = e^C \cdot e^{kt} \\
&\implies T - 5 = \pm e^C \cdot e^{kt} \\
&\implies T = 5 + Ae^{kt}
\end{aligned}
$$

where $A = \pm e^C$.

Use data to find A and k: We know that $T(0) = 100$ and $T(20) = 73$ (measuring time in minutes). From the first data point, we have

$$100 = T(0) = 5 + Ae^0 = 5 + A \implies A = 95.$$

From this and the second data point, we have

$$73 = T(20) = 5 + 95e^{20k} \implies 95e^{20k} = 68 \implies e^{20k} = \frac{68}{95}$$

so

$$k = \frac{1}{20}\ln(68/95) \qquad (\approx -0.01672)$$

Answer the question:

We want to find the time t_1 such that $T(t_1) = 20$, so we have to solve the equation

$$20 = T(t_1) = 5 + 95e^{kt_1}$$

Subtracting 5, dividing by 95 and taking logarithms gives

$$e^{kt_1} = \frac{15}{95} = \frac{3}{19} \implies kt_1 = \ln(3/19) \implies t_1 = 20 \cdot \frac{\ln(3/19)}{\ln(68/95)} \approx 176.12$$

I.e., the temperature of the ball will reach 20°C about 2 hours, 56 minutes and 7 seconds after the ball is dropped into the water.

Modeling population growth

1. *Exponential growth:* The simplest model for population growth is based on the assumption that the population grows *at a rate proportional to its size.* This leads to the differential equation

$$\frac{dP}{dt} = rP,$$

where $P(t)$ is the size of the population at time t, and r is an (unknown) parameter (called the intrinsic growth rate). This equation is easy to solve (after separating the variables):

$$\frac{dP}{P} = r\,dt \implies \int \frac{dP}{P} = r \int dt \implies \ln P = rt + C \implies P = Ae^{rt},$$

where

- $A = e^C$, and in fact...
- $A = P(0) = P_0$, the initial population size, and the *exponential growth* model is

$$P(t) = P_0 e^{rt}.$$

Example 5. The population of a small island in the year 1950 was 870 people, and in the year 2000, the population was 1250. Assuming exponential growth, what will the island's population be in the year 2050? How about in 2150?

Based on the assumption of exponential growth, we have

$$P(t) = 870e^{rt},$$

with time being measured in years, and $t = 0$ corresponding to the year 1950. This means that

$$\begin{aligned} 1250 = P(50) = 870e^{50r} &\implies e^{50r} = \frac{1250}{870} \\ &\implies 50r = \ln(125/87) \\ &\implies r = \frac{1}{50}\ln(125/87) \quad (\approx 0.00725) \end{aligned}$$

Therefore

$$P(100) = 870e^{100r} \approx 1796 \text{ and } P(200) = 870e^{200r} \approx 3708.$$

The exponential growth model can be quite accurate in the short run, but not in the long run... Because an exponentially growing population will eventually outstrip its resources.

2. *Logistic growth:* This model for population growth takes into account the fact that populations grow in environments that generally have a *carrying capacity*—a maximum sustainable size for the population growing there. The carrying capacity depends on the resources available in the environment.

The logistic model is based on the following assumptions/requirements.

(i) When the population is small (relative to the carrying capacity), it should grow at a rate (approximately) proportional to its size (like exponential growth).

(ii) As the population approaches the carrying capacity, the growth rate should approach 0.

(iii) The model should be as simple as possible.

These requirements lead to what is called the logistic equation:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right) = rP\left(\frac{M-P}{M}\right),$$

where r is the *intrinsic growth rate* and M is the carrying capacity.

The logistic equation is separable and is solved as follows.

$$\frac{dP}{dt} = rP\left(\frac{M-P}{M}\right) \implies \frac{dP}{P(M-P)} = \frac{r}{M}\,dt$$

$$\int \frac{dP}{P(M-P)} = \int \frac{r}{M}\,dt \implies \frac{1}{\cancel{M}}\ln\left|\frac{P}{M-P}\right| = \frac{rt}{\cancel{M}} + C$$

$$\implies \ln\left|\frac{P}{M-P}\right| = rt + C \implies \frac{P}{M-P} = Ae^{rt} \quad \text{where } A = \pm e^{C}$$

$$\implies P = (M-P)Ae^{rt} \implies P(1 + Ae^{rt}) = AMe^{rt}$$

$$\implies P = \frac{AMe^{rt}}{1 + Ae^{rt}} = \frac{M}{be^{-rt} + 1} \quad \text{where } b = A^{-1}$$

Another convenient way to express the logistic growth function uses the parameter $P_0 = P(0)$.

If $P(0) = P_0$, then using the equation $P = (M - P)Ae^{rt}$ and plugging in $t = 0$, gives

$$P_0 = A(M - P_0) \implies A = \frac{P_0}{M - P_0}.$$

Now substituting this into the expression

$$P = \frac{M}{A^{-1}e^{-rt} + 1}$$

gives

$$P(t) = \frac{M}{\frac{M-P_0}{P_0}e^{-rt} + 1} = \frac{MP_0}{P_0 + (M - P_0)e^{-rt}}$$

after multiplying top and bottom by P_0.