Differential equations.

A first order ordinary differential equation is an equation involving a function, its derivative and the free variable it depends on:

$$\Phi\left(y, \frac{dy}{dx}, x\right) = 0,$$

where it is assumed that $y = y(x)$.

*The equation is **separable** if it can be manipulated to look like this:*

$$h(y)\, dy = g(x)\, dx,$$

*i.e., if the variables (y and x in this case) can be **separated**.*

Example 1. The differential equation

$$\frac{y'}{x} - 3y = 0 \tag{1}$$

is separable because we can manipulate it as follows:

$$\frac{dy/dx}{x} - 3y = 0 \implies \frac{1}{x} \cdot \frac{dy}{dx} = 3y \implies \frac{1}{y}\, dy = 3x\, dx$$

A ***solution*** of a differential equation is a function $y = f(x)$ that satisfies the equation.

Example 1. (cont.) The function $y = e^{3x^2/2}$ is a solution of Eq. (1) ($y'/x - 3y = 0$) because $y' = 3xe^{3x^2/2}$ (chain rule), so

$$\frac{y'}{x} = \frac{3xe^{3x^2/2}}{x} = 3e^{3x^2/2} = 3y.$$

Observation: If A is any constant, then the function $y_A = Ae^{3x^2/2}$ is also a solution of Equation (1), because

$$y'_A = 3Axe^{3x^2/2} \implies \frac{y'_A}{x} = \frac{3Axe^{3x^2/2}}{x} = 3Ae^{3x^2/2} = 3y'_A.$$

Fact: If a differential equation has a solution, then it has infinitely many.

Why?

Because solving differential equations involves (indefinite) integration which entails an unknown constant of integration, leading to infinitely many different solutions.

Solving separable equations. To solve an equation of the form $h(y)\,dy = g(x)\,dx$... $\star$

(i) *Integrate both sides:*

$$h(y)\,dy = g(x)\,dx \implies \int h(y)\,dy = \int g(x)\,dx \implies H(y) = G(x) + C.$$

The resulting equation, $H(y) = G(x) + C$ is called the ***implicit solution***.

(ii) Solve the right hand equation for y: $\quad y = H^{-1}\big(G(x) + C\big)$. This is called the ***explicit solution*** (H^{-1} is the *inverse function* of H).

Example 1. (cont.) Integrating both sides of $\frac{1}{y}\,dy = 3x\,dx$ gives

$$\int \frac{1}{y}\,dy = \int 3x\,dx \implies \ln|y| = \frac{3x^2}{2} + C.$$

Next *exponentiate* both sides of the right hand equation (e^u is the inverse function of $\ln u$)

$$e^{\ln|y|} = e^{\frac{3x^2}{2}+C} \implies |y| = e^{(3x^2/2)+C} = e^C \cdot e^{3x^2/2} \implies y = Ae^{3x^2/2},$$

where $A = \pm e^C$.

Why differential equations?

*Mathematical models are based on theoretical assumptions about the variables in question. These assumptions are framed in terms of how the variables **change**, and change is described by the **derivatives** of the functions involved.*

Example: Newton's law of cooling (and heating) says:

> *The temperature T of a body immersed in a medium of constant temperature τ changes over time at a rate that is proportional to the difference $T - \tau$.*

The temperature of the body in this example is a function of time t, and Newton's law can be written as

$$\frac{dT}{dt} = k(T - \tau),$$

where k is the (generally unknown without more information) constant of proportionality.

Economic theory can also lead to differential equations...

Example 2. The *labor-elasticity of output* for a certain industry is assumed to be constant. Find the production function $q = f(l)$.

The elasticity of output q with respect to labor input l is defined as

$$\eta_{q/l} = \frac{dq}{dl} \cdot \frac{l}{q}.$$

The assumption of *constant elasticity* leads to the (separable) equation

$$\eta_{q/l} = \eta_0 \implies \frac{dq}{dl} \cdot \frac{l}{q} = \eta_0 \implies \frac{dq}{q} = \eta_0 \frac{dl}{l},$$

where η_0 is the (unknown) constant elasticity.

Integration gives the implicit solution

$$\int \frac{dq}{q} = \eta_0 \int \frac{dl}{l} \implies \ln q = \eta_0 \ln l + C.$$

Exponentiation gives the explicit solution

$$e^{\ln q} = e^{\eta_0 \ln l + C} \implies q = e^C \cdot e^{\ln(l^{\eta_0})} \implies q = A l^{\eta_0} \quad (\text{where } A = e^C)$$

To obtain a more precise mathematical model for the phenomenon being studied, scientists combine theoretical assumptions with ***data***. The data is used to find specific values for the (as-yet-unknown) parameters that appear in the (general) solutions of the differential equations.

Example 1. (cont.) We found that the (general) solution of the differential equation (1) is

$$y = Ae^{3x^2/2}.$$

To pick out a specific solution, we need to specify a value of the function at some point, called an *initial value*.

For example, if require that our solution $y = f(x)$ of (1) also satisfy $f(0) = 2$, then we find only one value of A that works:

$$2 = f(0) = Ae^{3 \cdot 0^2/2} = Ae^0 = A.$$

I.e., $y = 2e^{3x^2/2}$ is the *unique* solution of the ***initial value problem***

$$\frac{y'}{x} - 3y = 0; \quad y(0) = 2.$$